Statement of Research Interests – Dena Morton

The most basic objective of topology is that of the classification of spaces up to homeomorphism - in other words, the categorizing of topological types. Unfortunately, this task is an unreachable goal, so topologists instead try a coarser classification than homeomorphism, that of homotopy equivalence. Still, categorization up to homotopy equivalence is difficult to attain; the homotopy groups π_* come the closest to this objective with their classification of spaces up to weak homotopy equivalence. Homotopy groups, however, while of great importance in topology, often involve unwieldy (and many times impossible) calculations. Thus, for computational purposes one normally would use a different homotopy invariant, the homology groups H_* (and their counterparts, the cohomology groups H^*). These basic homology groups may be defined through the series of seven Eilenberg-Steenrod axioms. Calculatability has its trade-off, though; homology gives less information towards our goal than does homotopy.

Theories that satisfy the first six Eilenberg-Steenrod axioms are called generalized homology theories, and are often able to detect invariants which cannot be found by basic homology. Generalized homology theories include various K-theories, stable homotopy, and cobordism theories. Each generalized homology theory is represented by a spectrum (a generalization of a space). Spectra are indexed collections of spaces $\mathbf{X} = \{\underline{X}_n\}$ such that the suspension of the $\text{n}^{\underline{\text{th}}}$ space, $\sum \underline{X}_n$ is embedded as a subcomplex of $\underline{X}_{n+1}$, the $(\text{n+1})^{\underline{\text{st}}}$ space. The Brown representability theorem states the relationship between spectra and generalized homology theories:

Thm *Given a generalized homology theory $h_*(\ )$ and its corresponding cohomology theory $h^*(\ )$, there exists a spectrum $\mathbf{E}$ such that, for any space X, both of the following hold true:*

1. $h^*(X) = [X, \mathbf{E}]_*$, *the homotopy classes of maps from X to $\mathbf{E}$*
2. $h_*(X) = \pi_*(\mathbf{E} \wedge X)$, *the homotopy groups of $\mathbf{E}$ smash X*

Some examples of fundamental spectra and their corresponding generalized homology theories are

1. The Eilenberg-MacLane spectrum $H\mathbf{F}_2$, representing mod 2 homology, $H_*(\ ;\mathbf{Z}/2)$. This spectrum is composed of Eilenberg-MacLane spaces: $\underline{H\mathbf{F}_2}_n = K(\mathbf{Z}/2, n)$, with the property that each space has only one non-trivial homotopy group, namely $\pi_n(K(\mathbf{Z}/2, n)) = \mathbf{Z}/2$.
2. The real orthogonal K-theory spectrum KO, representing orthogonal K-theory. This spectrum begins with $\underline{KO}_0 = BO \times \mathbf{Z}$, where BO is the classifying space of the orthogonal group O. KO fulfills Bott periodicity: $\underline{KO}_n = \underline{KO}_{n+8}$.

3. The complex unitary K-theory spectrum KU, representing unitary K-theory. As with KO, this spectrum begins with $\underline{KU}_0 = BU \times \mathbf{Z}$. ($BU$ is the classifying space of the unitary group, U.) KU also fulfills Bott periodicity, but with a smaller cycle than KO: $\underline{KU}_n = \underline{KU}_{n+2}$.

4. The spectrum bo, the connected cover of KO, representing connective real K-theory. KO and bo are linked: $\underline{bo}_0 = BO \times \mathbf{Z} = \underline{KO}_0$. The spectrum bo is not periodic.

Each of the aforementioned spectra is an example of a loop spectrum – an infinite collection of spaces $\mathbf{X} = \{\underline{X}_n\}$ with the property that $\Omega(\underline{X}_{n+1})$ is homotopy equivalent to $\underline{X}_n$. (For a general space X, the loop space ΩX is the function space of all homotopy classes of loops of X , in other words, paths which both begin and end at the same point.)

In the 1970s, a new method to encode the deeper structure for the homology groups of loop spectra was presented in the shape of Hopf rings. The Hopf ring for the generalized homology theory of E, $H_*(\underline{E}_*)$, gives added insight into the product structure of that homology group, because, roughly speaking, it is a ring object in the category C_* of cogroups of the form $H_*(X;\mathbf{Z}/2)$. The Hopf ring structure includes a coproduct, two product structures, and relationships which mesh them together.

The first Hopf ring to be computed was for $H\mathbf{F}_2$, following the methods of J.P. Serre. Since then, many people have attended to these types of calculations. A main tool for these computations is the spectral sequence, similar to (but more complex than) exact sequences. After receiving input, a spectral sequence converges, with luck, to a desired homotopy or homology group. For loop spectra the bar spectral sequence is frequently used, since the bar spectral sequence converges to homology for the (n+1)$^{\underline{\text{st}}}$ space in a spectrum when it is given the homology for the n$^{\underline{\text{th}}}$ space.

The computation of Hopf rings adds to our understanding the structure of fundamental classes of spaces. For my part, I have been working with the spectra KO, KU and bo. In a paper entitled *The Hopf Ring for KO and KU* (joint with N. Strickland and submitted for publication), we recomputed the mod 2 homology for the spectra KO and KU as Hopf Rings. In this computation we made frequent use of the bar spectral sequence and of the properties of maps between the spectra KO and KU. My next paper, which is nearing completion, is entitled *The Hopf Ring for bo and its Connective Covers.* In this work I recompute the known ordinary mod 2 homology of the spectrum bo as a Hopf ring. There are two maps used in the computation of $H_*(bo)$:

1. A map of spectra, to the periodic Bott spectrum, $bo \to KO$.

2. A map of spectra, to the Eilenberg-MacLane spectrum, $bo \to H\mathbf{F}_2$.

Use of the Hopf ring properties for KO and $H\mathbf{F}_2$ allows, in conjunction with the maps above, for the establishment of the Hopf ring properties for elements in bo.

In the future, I hope to continue in this direction of study. It would be beneficial to know the Hopf ring structure for other spectra. I hope to work towards replacing $H_*(\mathbf{E})$, the homology of the spectrum $\mathbf{E}$, with $K(n)_*\mathbf{E}$, the Morava K-theory of the spectrum $\mathbf{E}$. Additionally, I am interested in computing the Hopf Ring for eo_2, the Hopkins-Miller real K-theory spectrum.