In section 1-2, #41 the following model for the amount owed M(t) owed on a mortgage after t years is considered:

$$\frac{dM}{dt} = iM - p,$$

where \(i\) is the annual interest rate (e.g. \(i=0.07\) when the interest rate is 7%), and where \(p\) is the annual payment. (Note: \(p\) is the total amount paid during a year, and the payment is assumed to occur "continuously", which is somewhat unrealistic. But only differentiable (thus continuous) processes can be described using a differential equation! Just think of paying \(\frac{p}{360}\) dollars each day. That will be close to paying "continuously".)

Additional information: Ms Lee wants a 20 year mortgage and there are two choices: borrow the money at 7% yearly interest (compounded continuously) or get an interest rate of 6.85% per year, if she pays $4500 up front, not counting towards the loan. (i.e. she pays $4500 to be able to borrow $150,000 at 6.85%.)

(a) How much does Ms Lee pay in each case?

Notice that in this problem, not all parameters in the model are known!!! Yes: \(p\) is not known. But it should be pretty clear that paying off the loan in exactly 20 years will occur for exactly one value of \(p\) !! In other words, the added initial condition that \(M(20) = 0\) must somehow, implicitly, determine the value of \(p\)!! Once we know \(p\), we know how much she pays each year, and therefore \(20p\) will be the total payments over 20 years. The other initial condition is, of course, \(M(0) = 150,000\).

So, not bothered by the fact that \(p\) is not known, and leaving the interest rate at "\(i\)" because we want to use different values later, we solve the differential equation, and then see where the initial conditions take us ...

Separating the variables we get

$$\int \frac{1}{iM - p} dM = \int 1 \, dt$$

and substituting \(u = iM - p\), \(du = i \, dM\) we obtain

$$\frac{1}{i} \int \frac{1}{u} \, du = t + C_1$$

$$\frac{1}{i} \ln|iM - p| = t + C_1$$

$$\ln|iM - p| = it + C_2$$

$$|iM - p| = e^{it + C_2} = C_3 e^{it}$$

$$iM = C_4 e^{it} + p$$
Now, notice that there are two unknowns: \( C_5 \) and \( p \). But we also have two initial conditions... and with two numbers to choose we will be able to satisfy both!

\[
M(t) = C_5 e^{it} + \frac{p}{i}
\]

and

\[
M(0) = 150,000 = C_5 e^0 + \frac{p}{i} = C_5 + \frac{p}{i}
\]

Since \( \frac{p}{i} = -C_5 e^{20i} \) (from the second equation) we have \( 150,000 = C_5 - C_5 e^{20i} \), leading to

\[
C_5 = \frac{150,000}{1 - e^{20i}}.
\]

So, \( \frac{p}{i} = -C_5 e^{20i} \) translates into \( \frac{p}{i} = -\frac{150,000}{1 - e^{20i}} \cdot e^{20i} \), so that

\[
p = -150,000 \cdot i \cdot \frac{e^{20i}}{1 - e^{20i}} = 150,000 \cdot i \cdot \frac{e^{20i}}{e^{20i} - 1},
\]

and factoring out and canceling \( e^{20i} \) we get

\[
p = \frac{150,000 \cdot i}{1 - e^{-20i}}.
\]

(a) How much does Ms Lee pay in each case?

If the interest is 7\%, \( i = 0.07 \), so that \( p \approx 13,936.76 \text{ \$ per year} \), and \( 20p \approx 278,735 \).

So, with the 7\% loan she would have total payments of $278,735 over the 20 years.

If the interest is 6.85\%, \( i=0.0685 \), so that \( p \approx 13,775.43 \text{ \$ per year} \), and \( 20p \approx 275,509 \).

So with the 6.85\% loan Ms. Lee would have total payments of $275,509 plus the $4500 up front, so that the loan would cost a total of 280,009.

(b) Which is a better deal over the entire time of the loan (assuming that she does not invest the $4500 in case she takes the 7\% loan)?

The 7\% loan is the better deal.

(c) If Ms. Lee can (and does) invest the $4500 at 5\% yearly interest compounded continuously (with the 7\% loan), which is the better deal?

Investing $4500 at 5\% for 20 years will lead to an account balance of $12,232 after 20 years. With the total payments of $278735 and $4500 invested, total payments would be $283,235, and the $12,232 would be hers, with a difference of $271,003 having left her pocket.
FINAL NOTE: Since we are dealing with a time span of twenty years, the value of a dollar changes significantly. A dollar in the beginning of the loan is worth more than a dollar at the end. In real purchasing power the numbers would be different and more useful for making a decision between different deals.